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TEST ELEMENTS FOR ENDOMORPHISMS OF FREE GROUPS AND ALGEBRAS

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ABSTRACT

There are two well-known approaches to recognizing automorphisms of a free group, i.e., to distinguishing automorphisms from non-automorphisms. The first one is the "inverse function theorem" of Birman. The second one, the "test element" approach, was originated by Nielsen for the free group of rank 2 and then extended to free groups of arbitrary finite rank by Zieschang, Rosenberger and others. In this note, we establish a direct connection between these two approaches: we associate a special matrix with any element of a free group, and show that an automorphism can be distinguished from a non-automorphism in terms of invertibility of such a matrix associated with a particular single element. Similar results hold for free associative and Lie algebras.

1. Introduction

Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a system $\{x_i\}$, $1 \leq i \leq n$, of free generators. By $\operatorname{Aut} F$ we denote the group of automorphisms, and by $\operatorname{End} F$ — the semigroup of endomorphisms of the group F.

We call $u \in F$ a **test element** if $\phi(u) = \alpha(u)$ for some $\phi \in \operatorname{End} F$ and $\alpha \in \operatorname{Aut} F$ implies that ϕ is actually an automorphism. In other words, we can distinguish an automorphism from a non-automorphism by means of its value on a single element, a test element. The condition $\phi(u) = \alpha(u)$ can be obviously replaced here with just $\phi(u) = u$. There are many test elements known by

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now; the first one has been exhibited by Nielsen [8] in the group F_2 — it is $u = [x_1, x_2]$. Then we can mention results of Zieschang [18], [19], Rosenberger [11], [12] ($u = x_1^m x_2^m \cdots x_n^m$, $m \geq 2$; $u = [x_1, x_2] \cdots [x_{n-1}, x_n]$ if n is even, and some others) and Rips [10] ($u = [x_1, \ldots, x_n]$). Dold [3] has found a series of test elements described in graph theoretic terms. Some necessary conditions for u to be a test element are given in [16]. Recently, Turner [17] has proved that an element of F is a test element if and only if it does not belong to any proper retract of F. However, there is still no effective procedure known for recognizing a test element.

In this note, we establish a direct connection between the "test element" approach to recognizing automorphisms and another approach, the so-called "inverse function theorem" due to Birman [1]. She proved the following: let ϕ be an endomorphism of the group F_n given by ϕ: $x_i \to y_i, 1 \leq i \leq n$. Define the matrix $J_\phi = \|d_j(y_i)\|_{1 \leq i,j \leq n}$ (the "Jacobian matrix" of ϕ), where d_j denotes Fox derivation in the free group ring $\mathbb{Z}F$ (see [4]). Then $y_1, \ldots, y_n$ generate the group F_n if and only if the matrix J_ϕ is invertible.

Here we define for any element $u \in F$, the "double Jacobian matrix" $D_u = \|d_i'(d_j(u))\|_{1 \leq i,j \leq n}$, where d_j is the "usual", or left Fox derivation, and d_i' is the right Fox derivation. Then we prove:

THEOREM 3.1: *Let ϕ be an endomorphism of the group F. It is an automorphism:*

(i) *if and only if the matrix $D_{\phi(u)}$ is invertible over $\mathbb{Z}F$ with $u = [x_1, x_2] \cdots [x_{n-1}, x_n]$, n even;*
(ii) *if and only if the natural image over Z_2F of the matrix $D_{\phi(u)}$ is invertible over Z_2F with $u = x_1^2 x_2^2 \cdots x_n^2$.*

A further goal of ours is to present elements with a stronger property than that just described; namely, we want to distinguish any two different endomorphisms of F by means of their value on a single element. There are no elements u with this property in the group F itself; in fact, we cannot even distinguish two automorphisms in this manner because every group element has non-trivial stabilizer in $\mathrm{Aut}\, F$. However, if we linearly extend an endomorphism of the group F to the group ring $\mathbb{Z}F$, it becomes possible to find such elements in $\mathbb{Z}F$; the simplest example is $u = \Sigma_{1 \leq i \leq n}\, p_i(x_i - 1)^2$ with different odd integers p_i (Proposition 3.5).

Some of these test elements from the group ring $\mathbb{Z}F$ have one more interesting property. Being in the image of an endomorphism of F (linearly extended to $\mathbb{Z}F$), they "enforce" this endomorphism to be an automorphism. More precisely: if for some $v \in \mathbb{Z}F$ one has $\phi(v) = u$, then ϕ is an automorphism. Again, there are obviously no elements with this property in the group F itself; however Rosenberger [11], [12] has proved that some elements of F (for example, $u = x_1^m x_2^m \cdots x_n^m, m \geq 2$) satisfy the following alternative: $u \in \phi(F)$ implies that either ϕ is an automorphism or u is a primitive element of the free group $\phi(F)$. This is the best possible result one can get considering test elements in the group F; but in the ring $\mathbb{Z}F$ we exhibit an element ($u = \Sigma_{1\leq i\leq n}(x_i - 1)^2$) for which only the first possibility in Rosenberger's alternative can occur (Theorem 3.3).

Finally, we note that a remarkable result of Turner [17] implies that if an element of F cannot be fixed by a non-monomorphism, then in fact it cannot be fixed by a non-automorphism.

The arrangement of the paper is as follows. In Section 2, we give necessary details on Fox calculus in the free group ring. In Section 3, we prove the results described above and then extend them to groups of the form $F/[R,R]$ (Proposition 3.6). In the concluding Section 4, we consider these issues for free associative and Lie algebras and get similar results.

2. Fox calculus

Let $\mathbb{Z}F$ be the integral group ring of the group F and Δ_F its augmentation ideal, that is, the kernel of the natural homomorphism σ: $\mathbb{Z}F \to \mathbb{Z}$. More generally, when $R \subseteq F$ is a normal subgroup of F, we denote by Δ_R the ideal of $\mathbb{Z}F$ generated by all elements of the form $(r-1)$, $r \in R$. It is the kernel of the natural homomorphism σ_R: $\mathbb{Z}F \to \mathbb{Z}(F/R)$.

In [4], Fox gave a detailed account of the differential calculus in a free group ring; we only recall a few things very briefly referring to [5] for more details.

The ideal Δ_F is a free left $\mathbb{Z}F$- module with a free basis $\{(x_i - 1)\}$, $1 \leq i \leq n$, and left Fox derivations d_i are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta_F$ can be uniquely written in the form $u = \Sigma_i d_i(u)(x_i - 1)$.

As the ideal Δ_F is a free right $\mathbb{Z}F$-module as well, one can define right Fox derivatives $d_i'(u)$ accordingly, so that $u = \Sigma_i(x_i - 1)d_i'(u)$.

One can extend these derivations linearly to the whole $\mathbb{Z}F$ defining $d_i'(1) =$

$d_i(1) = 0$.

The next lemma is an immediate consequence of the definitions.

LEMMA 2.1: *Let J be an arbitrary left (right) ideal of $\mathbb{Z}F$ and let $u \in \Delta_F$. Then $u \in \Delta_F J$ ($u \in J\Delta_F$) if and only if $d'_i(u) \in J$ ($d_i(u) \in J$) for each i, $1 \le i \le n$.*

We need the "chain rule" for left and right Fox derivations (cf. [4]):

LEMMA 2.2: *Let ϕ be an endomorphism of F (it can be linearly extended to $\mathbb{Z}F$) defined by $\phi(x_k) = y_k, 1 \le k \le n$, and let $v = \phi(u)$ for some $u, v \in \mathbb{Z}F$. Then:*

(a) $d_j(v) = \Sigma_{1 \le k \le n} \phi(d_k(u)) d_j(y_k)$;

(b) $d'_j(v) = \Sigma_{1 \le k \le n} d'_j(y_k) \phi(d'_k(u))$.

Remark 2.3: Since the augmentation ideal of any group ring $\mathbb{Z}_p F, p$ a prime, is a free left and right module over this ring, it is possible to consider Fox calculus in the rings $\mathbb{Z}_p F$ as well, and all technical results including Lemma 2.2 hold also in that situation.

3. Test elements in groups and group rings

THEOREM 3.1: *Let ϕ be an endomorphism of the group F. It is an automorphism:*

(i) *if and only if the matrix $D_{\phi(u)}$ is invertible over $\mathbb{Z}F$ with $u = [x_1, x_2] \cdots [x_{n-1}, x_n]$, n even;*

(ii) *if and only if the natural image over $\mathbb{Z}_2 F$ of the matrix $D_{\phi(u)}$ is invertible over $\mathbb{Z}_2 F$ with $u = x_1^2 x_2^2 \cdots x_n^2$.*

Proof: (i) Let $\phi(x_i) = y_i$, $1 \le i \le n$, $n = 2m$, and let $v = \phi(u)$. Apply a left Fox derivation d_j to both sides of this equality; by Lemma 2.2 (a), this gives

$$d_j(v) = \sum_{1 \le k \le n} \phi(d_k(u)) d_j(y_k). \tag{1}$$

Note that every $\phi(d_k(u))$ has augmentation 0 since u belongs to the commutator subgroup F', so that $d_k(u) \in \Delta_F, 1 \le k \le n$. Applying now a right Fox derivation d'_i to both sides of (1), we get by Lemma 2.2 (b):

$$d'_i(d_j(v)) = \sum_{1 \le k \le n} \sum_{1 \le m \le n} d'_i(y_m) \phi(d'_m(d_k(u))) d_j(y_k). \tag{2}$$

When i and j run through $\{1, \ldots, n\}$, (2) becomes a system of n^2 equalities which can be written in the matrix form as follows:

$$D_{\phi(u)} = J'_\phi \phi(D_u) J_\phi, \tag{3}$$

where $J_\phi = \|d_j(y_i)\|_{1 \le i,j \le n}$ is the left Jacobian matrix of ϕ; $J'_\phi = \|d'_i(y_j)\|_{1 \le i,j \le n}$ is the right Jacobian matrix of ϕ.

Suppose the matrix $D_{\phi(u)}$ is invertible. Then every matrix on the right hand side of (3) must be invertible too, which implies ϕ is an automorphism in view of the cited Birman's result.

Conversely, suppose ϕ is an automorphism. Then, again by Birman's theorem, the matrices J'_ϕ and J_ϕ are invertible (actually the result of Birman applies to the matrix J_ϕ, but applying Lemma 2.2 (b) immediately gives the same result for J'_ϕ as well). Now we have to consider the matrix D_u; it is a block-matrix having m 2×2 matrices B_k along the diagonal and zeros below; here

$$B_k = \begin{pmatrix} x_{2k-1}^{-1} x_{2k}^{-1}(x_{2k} - 1) & 1 - x_{2k-1}^{-1} x_{2k}^{-1}(x_{2k-1} - 1) \\ -x_{2k}^{-1} & x_{2k}^{-1} - x_{2k}^{-1} x_{2k-1} \end{pmatrix},$$

$1 \le k \le m$. It can be easily verified that every matrix B_k has the inverse

$$B_k^{-1} = \begin{pmatrix} x_{2k-1} - x_{2k-1}^2 & (1 - x_{2k-1})(x_{2k} - 1) - x_{2k} \\ x_{2k-1} & x_{2k} - 1 \end{pmatrix}.$$

Hence the matrix D_u is invertible, and so is the matrix $D_{\phi(u)}$ on the left-hand side of (3). This completes the proof of part (i) of the theorem.

(ii) The proof goes along the same lines as that of (i) after replacing the group ring $\mathbb{Z}F$ with $\mathbb{Z}_2F$. On the right-hand side of (1), the elements $\phi(d_k(u))$ will have augmentation 0 because, in the ring $\mathbb{Z}_2F$, one has $d_k(u) \in \Delta_F$ whenever $u \in F^2$. Then, Birman's theorem remains valid on replacing $\mathbb{Z}F$ with $\mathbb{Z}_2F$: in one direction, it follows from the "chain rule" in $\mathbb{Z}_2F$ (see Remark 2.3); in another direction, Birman's proof amounts to the following: if $y_1, \ldots, y_n, g$ are elements of F, and $(g-1)$ belongs to the right ideal of $\mathbb{Z}F$ generated by $(y_1 - 1), \ldots, (y_n - 1)$, then g belongs to the subgroup of F generated by $y_1, \ldots, y_n$. But this holds in an arbitrary group ring (see e.g. [9]). Finally, for $u = x_1^2 x_2^2 \cdots x_n^2$, the matrix D_u is an upper triangular matrix with the units on the diagonal; in particular, it is invertible over any ring, and this completes the proof.

It is easy to produce many examples of elements u of the group ring $\mathbb{Z}F$ with D_u invertible. However, in the group F itself, the only elements with this property I know are $[x_1, x_2] \cdots [x_{n-1}, x_n]$, $x_1^2 x_2^2 \cdots x_n^2$ and their automorphic images. I have no idea about what makes the double Jacobian matrix of an element $u \in F$ invertible. It is clear that if an element $u \in F$ has the matrix D_u invertible, then it is a test element for recognizing automorphisms; the converse, however, is not true. For example, elements of the form $x_1^p x_2^p \cdots x_n^p$, $p \geq 3$, being test elements, have double Jacobian matrix non-invertible.

Remark 3.2: If we switch left and right Fox derivatives in the definition of the double Jacobian matrix, i.e., put $D'_u = \|d_i(d'_j(u))\|_{1 \leq i \leq n}$, then D'_u appears to be the transpose of D_u. This follows from the "combinatorial" definition of the Fox derivatives (see [4]). In particular, Theorem 3.1 holds also on replacing $D_{\phi(u)}$ with $D'_{\phi(u)}$.

THEOREM 3.3: *Let ϕ be an endomorphism of the group F linearly extended to $\mathbb{Z}F$; $u = \sum_{1 \leq i \leq n} (x_i - 1)^2$, and let $u \in \phi(\mathbb{Z}F)$. Then ϕ is an automorphism.*

Proof: Let $u = \phi(v)$ for some $v \in \mathbb{Z}F$. We are going to prove that this implies $v \in \Delta_F^2$; then we can apply the argument from the proof of Theorem 3.1 because, for the given element u, the matrix D_u is just the the identity matrix, hence invertible. First we will prove it in the abelianized group ring $\mathbb{Z}(F/F')$ (we denote elements of $\mathbb{Z}F$ and their natural images in $\mathbb{Z}(F/F')$ by the same letters without ambiguity):

LEMMA 3.4: *Suppose in the group ring $\mathbb{Z}(F/F')$, one has $u = \phi(v)$ for some group endomorphism ϕ linearly extended to $\mathbb{Z}(F/F')$, and $u = \Sigma_{1 \leq i \leq n} (x_i - 1)^2$. Then $v \in \Delta^2$, Δ being the augmentation ideal of $\mathbb{Z}(F/F')$.*

Proof: Suppose $v = \Sigma c_i (x_i - 1) + w$, where $w \in \Delta^2$, and coefficients c_i are non-zero. Then

$$u = \sum c_i (\phi(x_i) - 1) + \phi(w). \tag{4}$$

Let $y_i = \phi(x_i)$; then (4) implies $\Sigma c_i (y_i - 1) \in \Delta^2$ which is possible only if $\Sigma c_i (y_i - 1) = 0$ in $\mathbb{Z}(F/F')$. Indeed, an obvious linear expansion shows that $\Sigma c_i (y_i - 1) = \Pi y_i^{c_i} - 1 \mod(\Delta^2)$, and if $\Pi y_i^{c_i} - 1 \in \Delta^2$ in $\mathbb{Z}(F/F')$, then $\Pi y_i^{c_i} = 1$ in the group F/F' — see e.g. [4]. Hence we have proved that (4) actually implies $u = \phi(w)$, and this completes the proof of Lemma 3.4.

To complete now the proof of the theorem, it is sufficient to consider the image of the equality $u = \phi(v)$ in the group ring $\mathbb{Z}(F/F')$ and apply Lemma 3.4.

Now we are going to show how one can distinguish any two different endomorphisms of F by means of their value on a single element of $\mathbb{Z}F$.

PROPOSITION 3.5: *Let $u = \Sigma_{1\le i\le n} p_i(x_i - 1)^2$ with different odd integers p_i, and let ϕ and ψ be two endomorphisms of F such that $\phi(u) = \psi(u)$. Then $\phi = \psi$.*

Proof: We need to prove the following: if for some elements $y_1, \ldots, y_n, z_1, \ldots, z_n$ of the group F, one has $\Sigma_{1\le i\le n} p_i(y_i - 1)^2 = \Sigma_{1\le i\le n} p_i(z_i - 1)^2$, then $y_i = z_i$, $1 \le i \le n$. This will follow from another statement: if in the group ring Z_2F, one has $\Sigma_{1\le i\le k} c_i(y_i - 1)^2 = 0$ for some $c_i \in Z_2$ and pairwise distinct $y_i \in F$, then all coefficients c_i are zero.

Assume that we have $\Sigma_{1\le i\le m} c_i(y_i - 1)^2 = 0$ for some $m \ge 2$, non-trivial y_i, and non-zero $c_i \in Z_2$, $1 \le i \le m$. This means that between y_i, there are relations of the form $y_j^2 = y_k^2$, $j \ne k$. This implies $y_j = y_k$ which contradicts the assumption about y_i. The result follows.

To conclude this section, we note that some of our results can be extended to a more general situation of groups of the form F/R', R being an arbitrary normal subgroup of F, and $R' = [R, R]$ — its commutator subgroup. Here we will consider generating systems instead of automorphisms to avoid restrictions on a group, like being Hopfian etc.

We are going to use the following facts:

(a) a set of elements generates the group F modulo R' if and only if it generates F modulo $[R', F]$;

(b) if $g \in [R', F]$, then $g - 1 \in \Delta_F \Delta_R \Delta_F$ — see e.g. [5, p. 113];

(c) elements $y_1, \ldots, y_n$ generate the group F modulo R' if and only if the matrix $\|\sigma_R(d_j(y_i))\|_{1\le i,j\le n}$ is invertible over the ring $\mathbb{Z}(F/R)$ — see [7].

Now we can come up with

PROPOSITION 3.6: *Elements $y_1, \ldots, y_n$ generate the group F modulo R':*

(i) *if and only if the matrix $\sigma_R(D_{\phi(u)})$ is invertible over $\mathbb{Z}(F/R)$ with $u = [x_1, x_2] \cdots [x_{n-1}, x_n]$, n even;*

(ii) *if and only if the natural image over $\mathbb{Z}_2(F/R)$ of the matrix $\sigma_R(D_{\phi(u)})$ is invertible over $\mathbb{Z}_2(F/R)$ with $u = x_1^2 x_x^2 \cdots x_n^2$.*

Proof: This goes along the same lines as that of Theorem 3.1, but we consider equality $v = \phi(u)$ modulo $\Delta_F \Delta_R \Delta_F$ (here we use (a) and (b)); equality (1) —

modulo $\Delta_F \Delta_R$ (by Lemma 2.1), and equalities (2) and (3) — modulo Δ_R and $M_n(\Delta_R)$ respectively. Then we use Krasnikov's result (c) instead of Birman's to complete the proof.

We note that a similar idea has been used in [6] while proving the "if" part of Nielsen's commutator test for two-generator groups of the form $F/[R', F]$.

4. Tests elements in associative and Lie algebras

Similar to the free groups situation, the first test element for automorphisms of algebras was discovered in the free associative algebra A_2 of rank 2 — see [2]. We denote here by A_n the free associative algebra of rank $n \geq 2$ with a fixed set $\{x_i\}$, $1 \leq i \leq n$, of free generators. By L_n we denote the free Lie algebra with the same generating set, and consider it naturally embedded into A_n. Both A_n and L_n are considered over the same field of characteristic zero.

Fox calculus can be used in the algebras A_n in a natural way — see [14]. In [15], it has been used for proving the following characterization of **homogeneous** test elements of Lie algebra L_n: a homogeneous element of L_n is a test element if and only if it does not belong to any proper free factor of L_n (in other words, any of its automorphic images depends on exactly n free generators). Furthermore, based on the description of generating systems for $L_n/[R, R]$ algebras [14] (R being an arbitrary ideal of L_n) similar to Krasnikov's result for groups [7], we can prove the following analog of Proposition 3.5 (we use the obvious parallelism in the notation; in particular, Δ_R denotes the ideal of A_n generated by elements from R, and it is the kernel of the natural homomorphism $\sigma_R : L_n \to L_n/R$ extended to the universal enveloping algebras):

PROPOSITION 4.1: *Elements $y_1, \ldots, y_n$ generate the algebra L_n modulo $[R, R]$ if and only if the matrix $\sigma_R(D_{\phi(u)})$ is invertible over A_n/Δ_R with*

(i) $u = [x_1, x_2] + [x_3, x_4] + \cdots + [x_{n-1}, x_n]$, *$n$ even;*
(ii) $u = [x_1, x_2] + [x_2, x_3] + \cdots + [x_{n-1}, x_n]$.

Here $[a, b] = ab - ba$, Lie commutator of elements a and b.

It is easy to see that the matrix D_u in both cases in invertible, so the argument from the proof of Theorem 3.1 (i) works in this situation as well.

To characterize test elements of a free associative algebra is a more difficult problem because here we don't have the facility of using the "inverse function theorem" like in a free group [1] or in a free Lie algebra [14] situation. However, in the case of A_2, the free associative algebra of rank 2, we do have something similar — see [2], [13]. This enables us, for example, to find stabilizers in Aut A_2 of some particular elements from A_2 by using our method:

PROPOSITION 4.2: *An automorphism ϕ of the algebra A_2 fixes the element $u = x_1x_2 + x_2x_1$ if and only if either $\phi(x_1) = \alpha x_1, \phi(x_2) = (1/\alpha)x_2$, or $\phi(x_1) = \alpha x_2, \phi(x_2) = (1/\alpha)x_1$ for some non-zero $\alpha \in K$.*

Proof: Proceeding as in the proof of Theorem 3.1 (i), we arrive at

$$D_u = J'_\phi \phi(D_u) J_\phi,$$

where $\phi(D_u) = D_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $J'_\phi B = I$, the identity matrix; here

$$B = \begin{pmatrix} d_2(y_2) & d_2(y_1) \\ d_1(y_2) & d_1(y_1) \end{pmatrix}.$$

On the other hand, by the result of [13] we have $J'_\phi B' = \beta I$, where

$$B' = \begin{pmatrix} d_2(y_2) & -d_2(y_1) \\ -d_1(y_2) & d_1(y_1) \end{pmatrix},$$

and $\beta \in K$, a non-zero element. This implies the following set of alternatives:

(1) $d_2(y_1) = 0$ or $d'_1(y_1) = 0$;

(2) $d_1(y_2) = 0$ or $d'_2(y_2) = 0$;

(3) $d_2(y_1) = \alpha$ or $d'_1(y_2) = \alpha$ for an appropriate $\alpha \in K$;

(4) $d_1(y_2) = \alpha$ or $d'_2(y_1) = \alpha$ for an appropriate $\alpha \in K$.

Considering all possible combinations yields the result; we omit the details.

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